A Generalized Solution for Diversity Combining Techniques in Fading Channels

Claudio R. C. M. da Silva and Michel Daoud Yacoub

***Abstract*—This paper develops an *exact* and *general* formulation in order to investigate the performance of equal gain, selection/equal gain, and selection/maximal ratio combining techniques. The performance is examined in terms of both the reliability and mean signal-to-noise ratio at the combiner output. Although the solution presented here considers the Nakagami fading environment, the formulation is general and can be easily applied to any other fading channel.**

***Index Terms*—Diversity combining techniques, Nakagami fading channels, Rice fading channels.**

I. Introduction

The performance of wireless systems is severely affected by fading. Diversity combining techniques have long been used in wireless systems in order to minimize the degradation effects due to fading. Among the *conventional combining techniques* the pure selection combining (PSC), equal gain combining (EGC), and maximal ratio combining (MRC) have been intensively explored. More recently, hybrid techniques such as selection/maximal ratio combining (SMC), which associates PSC and MRC, and selection/equal gain combining (SEC), which associates PSC and EGC, have been proposed [1]. Despite all the intensive investigations, to the best of our knowledge, the performance of EGC and SEC has been confined to applications in which the Rayleigh channel is the focus. In addition, the performance analysis of SMC and SEC has been restricted to *particular* solutions in which a very limited number of branches is tackled [1], [2].

This paper develops an *exact* and *general* formulation in order to investigate the performance of EGC, SMC, and SEC. The performance is examined in terms of the reliability (the complement of the distribution) as a function of the signal-to-noise ratio (SNR) and the mean SNR at the combiner output. Although the solution presented here considers the Nakagami fading environment, the formulation can be easily applied to any other fading channel. The investigations assume equal noise power N at all L branches.

II. Nakagami-m Distribution

The Nakagami-m probability density function $p(\alpha_i)$ of the normalized SNR $\alpha_i = \gamma_i/\gamma_0$ at branch i, where γ_i is the SNR at branch i and γ_0 is the mean SNR, assumed identical for all branches, is given by

$$p(\alpha_i) = \frac{m^m}{\Gamma(m)} \alpha_i^{m-1} \exp(-m\alpha_i) \tag{1}$$

where $m \geq 1/2$ is the Nakagami fading parameter and $\Gamma(m)$ is the Gamma function. The corresponding probability distribution function $P(\alpha_i)$ is

$$P(\alpha_i) = \frac{\Gamma(m, m\alpha_i)}{\Gamma(m)} \tag{2}$$

where $\Gamma(a,b) = \int_0^b x^{a-1} \exp(-x)\, dx$ is the incomplete Gamma function. In our analysis, we shall also make use of $p(u_i)$, the probability density function of the normalized envelope $u_i = r_i/\sqrt{w_0}$ of branch i, where r_i is the envelope at branch i and w_0 is its mean power, assumed identical for all branches. Knowing that $\gamma_i = r_i^2/2N$ and that $\gamma_0 = w_0/N$

$$p(u_i) = \frac{m^m}{2^{m-1}\Gamma(m)} u_i^{2m-1} \exp\left(-\frac{mu_i^2}{2}\right). \tag{3}$$

The corresponding probability distribution function $P(u_i)$ is

$$P(u_i) = \frac{\Gamma\left(m, \frac{mu_i^2}{2}\right)}{\Gamma(m)}. \tag{4}$$

III. Combining Techniques

In PSC, the best signal is always present at the output of the combiner. In EGC, the signals of the L branches are co-phased and added to yield the output. In MRC, the signals of the L branches are co-phased, conveniently amplified, and added to yield the output. In SMC, the conventional PSC is followed by a MRC such that, out of the L branches, PSC chooses the l largest signals and MRC performs the optimal weighted sum of the l co-phased signals. In this case, channel estimation is performed only for the l selected signals, whereas in MRC this must be carried out for all of the L signals. In SEC, the conventional PSC is followed by a EGC such that, out of the L branches, PSC chooses the l largest signals and EGC performs the sum of the l co-phased signals. Note that, in this case, the channel estimation is completely eliminated as in EGC; hence, simplifying the whole circuitry. For the calculations that follow, we define $\alpha = \gamma/\gamma_0$, where γ is the combiner output SNR.

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IV. Reliability—Conventional Techniques

A. PSC

In PSC, with the best signal always present at the output, the probability that the SNRs at all of the L branches are simultaneously less than or equal to a given α is

$$P_{\mathrm{PSC}}(\alpha) = \mathrm{prob}(\alpha_1, \alpha_2, \ldots, \alpha_L \le \alpha) = \left[\frac{\Gamma(m, m\alpha)}{\Gamma(m)}\right]^L. \tag{5}$$

B. EGC

In EGC, the received signals with envelopes r_i, $i = 1, 2, \ldots, L$, are co-phased and added so that the combiner output envelope r is $r = \sum_{i=1}^{L} r_i$. The corresponding SNR is

$$\gamma = \frac{r^2}{2NL} = \frac{\left(\sum_{i=1}^{L} r_i\right)^2}{2NL}. \tag{6}$$

The density $p(\gamma)$ of γ is found as

$$p(\gamma) = NL\frac{p(r)}{r} = NL\frac{p\left(\sqrt{2NL\gamma}\right)}{\sqrt{2NL\gamma}} \tag{7}$$

which is the density of a sum of independent fading envelopes. The question of finding such a density for wireless channels dates back to Lord Rayleigh, for the Rayleigh fading case, and no closed-form solution is available. Brennan [3] has tackled this, for the Rayleigh fading case, by means of numerical methods based on an L-fold convolution of the densities. Following Brennan [3], and after a lengthy and tedious procedure, the distribution of α, now for the Nakagami environment, is found to be

$$P_{\mathrm{EGC}}(\alpha) = \int_0^a \int_0^{a-u_L} \cdots \int_0^{a-\sum_{i=3}^{L} u_i} \int_0^{a-\sum_{i=2}^{L} u_i} \prod_{i=1}^{L} p(u_i) du_i \tag{8}$$

where $a = \sqrt{2L\alpha}$ and $p(u_i)$ is the probability density function of the normalized envelope, as given in (3).

C. MRC

In MRC, the signals are co-phased and conveniently weighted so that the resultant SNR is maximized. In such a case, the normalized output SNR α is given by the sum of the L individual SNR's α_i as

$$\alpha = \sum_{i=1}^{L} \alpha_i. \tag{9}$$

The probability density function $p_{\mathrm{MRC}}(\alpha)$ of α is obtained as a result of an L-fold convolution of the individual densities of α_i given by (1) so that

$$p_{\mathrm{MRC}}(\alpha) = \frac{m^{mL}}{\Gamma(mL)} \alpha^{mL-1} \exp(-m\alpha). \tag{10}$$

The corresponding probability distribution function is

$$P_{\mathrm{MRC}}(\alpha) = \frac{\Gamma(mL, m\alpha)}{\Gamma(mL)}. \tag{11}$$

V. Reliability—Hybrid Techniques

The notation (L, l) is used to indicate the number of selected branches (l) out of the total number of branches (L).

A. SMC

In SMC, l out of L input signals are chosen that present the largest SNRs. The selected signals are then added in an optimum manner with the resultant normalized SNR given by $\alpha = \sum_{i=1}^{l} \alpha_i$. The selection process results in a set of dependent random variables with joint probability density function obtained as [4]

$$p(\alpha_1, \ldots, \alpha_l) = l! \binom{L}{l} [P(\alpha_l)]^{L-l} \prod_{i=1}^{l} p(\alpha_i) \tag{12}$$

where $P(\alpha_l)$ is given by (2) with l replacing i, $p(\alpha_i)$ is given by (1), and $\alpha_1 \ge \ldots \ge \alpha_l \ge 0$.

The restriction $\alpha_1 \ge \ldots \ge \alpha_l \ge 0$ imposes dependence among the variables, and the simple l-fold convolution process used to obtain the resultant density, as in MRC, no longer applies. The distribution now is obtained by integrating the density (12) over the volume defined by $\alpha_1 \ge \ldots \ge \alpha_l \ge 0$ and $\alpha = \sum_{i=1}^{l} \alpha_i$. We generalize the result of [1] and write the resultant distribution as

$$P_{\mathrm{SMC}}(\alpha) = \int_0^{\alpha/l} \int_{\alpha_l}^{(\alpha-\alpha_l)/(l-1)} \int_{\alpha_{l-1}}^{(\alpha-\alpha_l-\alpha_{l-1})/(l-2)} \cdots \int_{\alpha_3}^{\left(\alpha-\sum_{i=3}^{l} \alpha_i\right)/2} \int_{\alpha_2}^{\alpha-\sum_{i=2}^{l} \alpha_i} p(\alpha_1, \ldots, \alpha_l) \times d\alpha_1 \ldots d\alpha_l \tag{13}$$

where the joint density is given by (12).

It is noteworthy that, for the particular case in which $l = 1$, the probability distribution function given by (13) reduces to that of PSC, given by (5). In the same way, for $L = l$, the probability distribution function given by (13) reduces to that of MRC, given by (11).

B. SEC

In SEC, l out of L input signals are chosen that present the largest SNRs. The selected signals are then added as in EGC so that, at the combiner output, the resultant normalized envelope is $u = \sum_{i=1}^{l} u_i$. The joint probability density function of the selected normalized signals is [4]

$$p(u_1, \ldots, u_l) = l! \binom{L}{l} [P(u_l)]^{L-l} \prod_{i=1}^{l} p(u_i) \tag{14}$$

where $P(u_l)$ is given by (4) with l replacing i, $p(u_i)$ is given by (3), and $u_1 \ge \ldots \ge u_l \ge 0$. The relationship between the normalized envelope of the resultant signal and its SNR can be conveniently obtained from the definition of the normalized envelope and from (6), now for l signals, as

$$u = \sqrt{2l\alpha}. \tag{15}$$

As in the SMC case, the restriction $u_1 \geq \ldots \geq u_l \geq 0$ imposes dependence among the variables, and the distribution of the normalized envelope is obtained by integrating the density (14) over the volume defined by $u_1 \geq \ldots \geq u_l \geq 0$ and $u = \sum_{i=1}^{l} u_i$. Given the distribution of u and knowing that (15) holds, the distribution of α is

$$P_{\mathrm{SEC}}(\alpha) = \int_0^{u/l} \int_{u_l}^{(u-u_l)/(l-1)} \int_{u_{l-1}}^{(u-u_l-u_{l-1})/(l-2)} \cdots \int_{u_3}^{\left(u-\sum_{i=3}^{l} u_i\right)/2} \int_{u_2}^{u-\sum_{i=2}^{l} u_i} p(u_1, \ldots, u_l) \times du_1 \ldots du_l \tag{16}$$

where the joint density function is given by (14) and u is a function of α, as given in (15). For the particular cases in which $l = 1$, the probability distribution function given by (16) reduces to that of PSC, given by (5). In the same way, for $L = l$, the probability distribution function given by (16) reduces to that of EGC, given by (8).

VI. Mean SNR—Conventional Techniques

Another performance parameter of interest is the mean SNR $\bar{\gamma}$, which in its normalized form $\bar{\alpha}$ is given as $\bar{\alpha} = \bar{\gamma}/\gamma_0$.

A. PSC

A standard procedure for the calculation of the mean value of a random variable given its probability distribution function has been used [4]. An intricate manipulation of the resulting expression yields

$$\bar{\alpha} = \sum_{i=1}^{L} \overline{\alpha^i} \tag{17}$$

where $\overline{\alpha^i} = \int_0^\infty [P(\alpha)]^{i-1} [1 - P(\alpha)] d\alpha$ and $P(\alpha)$ is given by (2) with the subscript dropped. For $L = 2$, (17) reduces to (18) of [5] so that

$$\bar{\alpha} = 1 + \frac{\Gamma(2m)}{2^{2m-1}\Gamma(m)\Gamma(m+1)}. \tag{18}$$

B. EGC

The mean SNR is obtained from (6) as

$$\bar{\gamma} = E(\gamma) = \frac{1}{2NL} E\left[\left(\sum_{i=1}^{L} r_i\right)^2\right] = \frac{1}{2NL} \sum_{i,j=1}^{L} E(r_i r_j). \tag{19}$$

It can be seen from (19) that there are L elements equal to $E\left(r_i^2\right)$ and $L(L-1)$ elements equal to $E(r_i r_j)$, $i \neq j$. For the Nakagami-m distribution, $E\left(r_i^2\right) = 2\gamma_0 N$ and $E(r_i) = \Gamma(m+0.5)\sqrt{2\gamma_0 N/m}/\Gamma(m)$. Therefore,

$$\bar{\alpha} = 1 + \frac{L-1}{m}\left(\frac{\Gamma(m+0.5)}{\Gamma(m)}\right)^2. \tag{20}$$

C. MRC

The mean SNR is given by $\bar{\gamma} = \sum_{i=1}^{L} E(\gamma_i)$ yielding

$$\bar{\alpha} = L. \tag{21}$$

VII. Mean SNR—Hybrid Techniques

A standard procedure for the calculation of the mean value of a random variable given its probability density function [4] has been used for both hybrid techniques. In such a case, $\bar{\alpha} = \int_0^\infty \alpha p(\alpha) d\alpha$, where the density function $p(\alpha)$ is found from the distribution function, i.e., (13) for SMC and (16) for SEC. Due to the complexity of the resulting expressions, the upper limit of the integral (infinite) renders the calculations of difficult convergence. This problem is circumvented by a change of variable such as $h = \exp(-\alpha)$, for which the limits become zero and one.

A. SMC

Following the procedure as described above, we find

$$\bar{\alpha} = l!\binom{L}{l}\left(\frac{m^m}{\Gamma(m)}\right)^l \int_0^1 \int_0^{\hat{h}/l} \int_{\alpha_l}^{(\hat{h}-\alpha_l)/(l-1)} \times \int_{\alpha_{l-1}}^{(\hat{h}-\alpha_l-\alpha_{l-1})/(l-2)} \cdots \int_{\alpha_3}^{\left(\hat{h}-\sum_{i=3}^{l}\alpha_i\right)/2} \hat{h}\,[P(\alpha_l)]^{L-l} \times (h\alpha_2 \ldots \alpha_l)^{m-1} \left(\hat{h} - \sum_{i=2}^{l} \alpha_i\right)^{m-1} d\alpha_2 \ldots d\alpha_l dh \tag{22}$$

where the distribution is given by (2), with l replacing i, and $\hat{h} = -\ln(h)$. For the Rayleigh case, (22) reduces to [6]

$$\bar{\alpha} = \left(1 + \sum_{i=l+1}^{L} \frac{1}{i}\right) l. \tag{23}$$

B. SEC

Following the procedure as previously described above, we find

$$\bar{\alpha} = \int_0^1 \int_0^{a/l} \int_{u_l}^{(a-u_l)/(l-1)} \int_{u_{l-1}}^{(a-u_l-u_{l-1})/(l-2)} \cdots \int_{u_3}^{\left(a-\sum_{i=3}^{l} u_i\right)/2} \sqrt{\frac{-l\ln(h)}{2h^2}} \times p\left(\left(u_1 = a - \sum_{i=2}^{l} u_i\right), \ldots, u_l\right) du_2 \ldots du_l dh \tag{24}$$

where $a = \sqrt{-2l\ln(h)}$ and the joint density is given by (14).

VIII. FURTHER EXAMPLE—RICE MODEL

The analysis developed here can be easily extended to other fading models, such as the Rician model. In such a case, the probability density function $p(\alpha_i)$ of the normalized SNR $\alpha_i = \gamma_i/\gamma_0$ at branch i is given by

$$p\left(\alpha_i\right) = (1+k)\exp\left(-k-(1+k)\,\alpha_i\right) I_0\left(2\sqrt{k\,(1+k)\,\alpha_i}\right) \tag{25}$$

where $k \geq 0$ is the Rice parameter, and the corresponding probability distribution function $P(\alpha_i)$ is given by

$$P\left(\alpha_i\right) = 1 - Q\left(\sqrt{2k}, \sqrt{2\,(1+k)\,\alpha_i}\right) \tag{26}$$

where $Q\left(a,b\right) = \int_b^\infty x \exp\left(-\left(x^2+a^2\right)/2\right) I_0\left(ax\right) dx$ is the Marcum-Q function. The reliability of the SMC in a Rician environment is given by (13), with the normalized functions in (12) given by (25) and (26). In the same way, the analysis developed for the EGC and SEC can be extended to the Rician environment, in which case the statistics of the normalized envelope $u_i = r_i/\sqrt{w_0}$ of the Rice model is used. The probability density function $p(u_i)$ in this case is given by

$$p\left(u_i\right) = (1+k)\,u_i \exp\left(-k - \frac{(1+k)\,u_i^2}{2}\right) \times I_0\left(\sqrt{2k\,(1+k)}u_i\right) \tag{27}$$

and the probability distribution function $P(u_i)$ is

$$P(u_i) = 1 - Q\left(\sqrt{2k}, \sqrt{(1+k)}u_i\right). \tag{28}$$

The reliability of the EGC can be obtained with (27) in (8). The reliability and mean SNR of the SEC are, respectively, supplied by (16) and (24) with the normalized functions in (14) being given by (27) and (28).

IX. RESULTS

The performances of EGC and MRC are compared in Fig. 1, where their reliabilities are plotted for $m = 0.5$ and $L = 1, 2, 3, 4$, and 5 branches, and for $m = 2$ and $L = 2, 3, 4$, and 5 branches (the case $L = 1$ is not shown for the sake of clarity). It can be seen that both combining techniques yield similar performances: 1) for a small number of branches and 2) for less severe fading conditions. We observe that, for both techniques, and for a given reliability, the gains obtained for the extreme fading condition $(m = 0.5)$ are much better than those obtained for a better fading situation $(m = 2)$, and the gain increases with the increasing number of branches.

Fig. 2 depicts the reliability of SMC for $L = 5$ and $l = 1, 2, 3, 4$, and 5 both for $m = 0.5$ and $m = 2$. We note that, the greater the number of selected branches, the better the performance; this being more noticeable for better fading conditions.

Fig. 3 shows the reliability of SEC for $L = 5$ and $l = 1, 2, 3, 4$, and 5. Here, an intriguing behavior can be observed. As opposed to what happens in SMC, the increase of the number of selected branches does not directly imply an improvement of the performance, this greatly depending on the fading conditions. For instance, for $m = 0.5$, the best performance is achieved for $l = 3$, the performance degrading for $l = 4$, $l = 2$, $l = 5$, and $l = 1$, in this order. For $m = 2$, on the other hand, the performance improves as the number of selected branches increases.

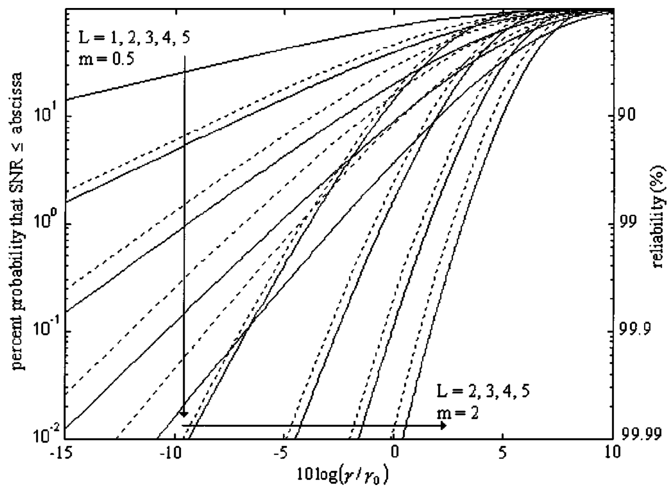

Fig. 1. Distribution of the SNR of EGC (dotted line) and MRC (solid line).

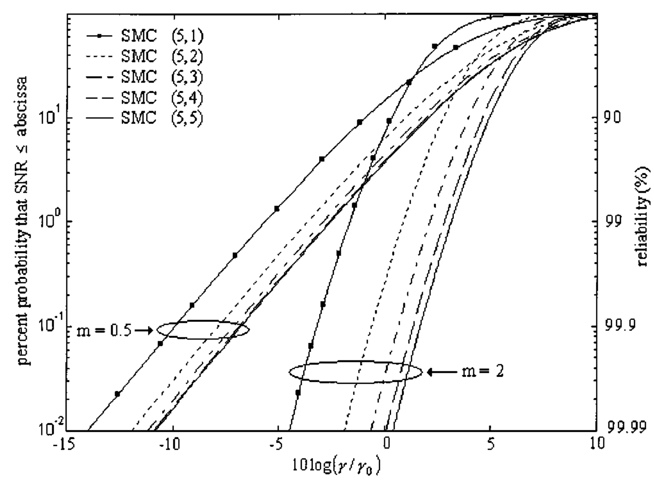

Fig. 2. Distribution of the SNR for SMC.

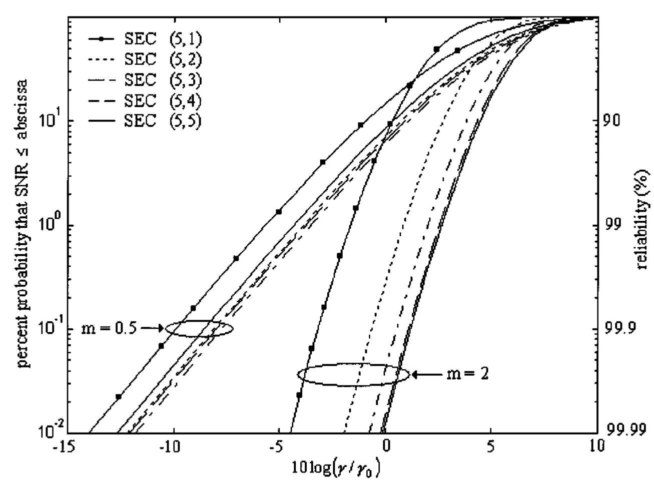

Fig. 3. Distribution of the SNR for SEC.

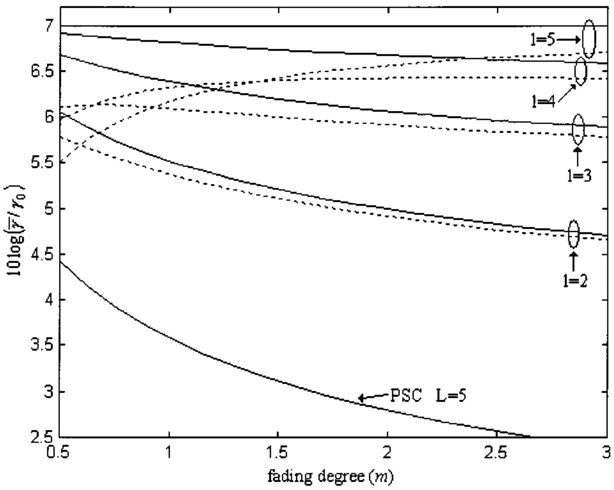

Fig. 4. Mean SNR for SEC (dotted line) and SMC (solid line); $L = 5$.

This phenomenon can be explained observing (6). As the number of chosen signal increases, the resultant envelope is enhanced, but the noise component also increases. For a bad fading condition in an nonoptimum addition, as in EGC, the inclusion of the signal of one more branch may not be enough to counterbalance the addition of the noise present at that branch.

Fig. 4 shows the normalized mean SNR as a function of the Nakagami fading parameter for both SMC and SEC with $L = 5$ and $l = 1, 2, 3, 4$, and 5. Note that SMC always performs better than SEC for the same number of selected branches. Note also that, as the fading conditions improve and for the same number of selected branches, the performance of SEC approaches that of SMC. The performance of SMC always improves with the increase of the number of selected branches independently of the fading conditions. On the other hand, the performance of SEC is greatly dependent on the fading environment, as can be observed by the crossings of the curves along the m-axis in Fig. 4. In general, for SEC, it can be said that, as the fading condition deteriorates, a better performance is obtained if the number of selected branches diminishes. Therefore, given a fading condition, the number of selected branches can be found that yields a better performance.

X. Conclusions

This paper has developed an *exact* and *general* formulation in order to investigate the performance of equal gain, selection/equal gain, and SMC techniques, given in terms of reliability and mean SNR. It was shown that the simple EGC may yield a performance close to that obtained by the optimum technique, namely MRC. This paper has shown that the performance of SMC always improves with the increase of the number of selected branches, independently of the fading environment, while the performance of SEC, on the contrary, is greatly dependent on the fading conditions. The number of selected branches in SEC can be chosen as the fading conditions vary in order to yield a better performance. The solutions were illustrated for some cases, but the formulation is general and can be easily applied to different fading channels.

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